4

Plane stress and plane strain

4.1 Introduction

Two-dimensional elastic problems were the first successful examples of the application of the finite element method.[1,2] Indeed, we have already used this situation to illustrate the basis of the finite element formulation in Chapter 2 where the general relationships were derived. These basic relationships are given in Eqs (2.1)–(2.5) and (2.23) and (2.24), which for quick reference are summarized in Appendix C.

In this chapter the particular relationships for the plane stress and plane strain problem will be derived in more detail, and illustrated by suitable practical examples, a procedure that will be followed throughout the remainder of the book.

Only the simplest, triangular, element will be discussed in detail but the basic approach is general. More elaborate elements to be discussed in Chapters 8 and 9 could be introduced to the same problem in an identical manner.

The reader not familiar with the applicable basic definitions of elasticity is referred to elementary texts on the subject, in particular to the text by Timoshenko and Goodier,[3] whose notation will be widely used here.

In both problems of plane stress and plane strain the displacement field is uniquely given by the u and v displacement in the directions of the cartesian, orthogonal x and y axes.

Again, in both, the only strains and stresses that have to be considered are the three components in the xy plane. In the case of *plane stress*, by definition, all other components of stress are zero and therefore give no contribution to internal work. In *plane strain* the stress in a direction perpendicular to the xy plane is not zero. However, by definition, the strain in that direction is zero, and therefore no contribution to internal work is made by this stress, which can in fact be explicitly evaluated from the three main stress components, if desired, at the end of all computations.

4.2 Element characteristics

4.2.1 Displacement functions

Figure 4.1 shows the typical triangular element considered, with nodes i, j, m

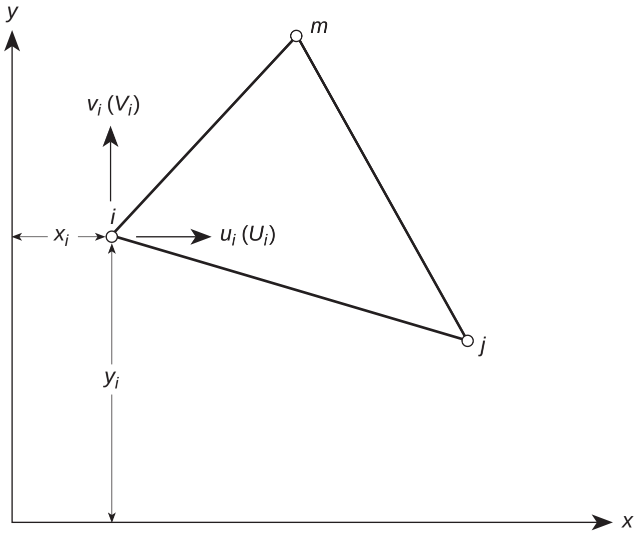


Fig. 4.1 An element of a continuum in plane stress or plane strain.

numbered in an anticlockwise order. The displacements of a node have two components

$$\mathbf{a}_i = \begin{Bmatrix} u_i \\ v_i \end{Bmatrix} \tag{4.1}$$

and the six components of element displacements are listed as a vector

$$\mathbf{a}^e = \begin{Bmatrix} \mathbf{a}_i \\ \mathbf{a}_j \\ \mathbf{a}_m \end{Bmatrix} \tag{4.2}$$

The displacements within an element have to be uniquely defined by these six values. The simplest representation is clearly given by two linear polynomials

$$\begin{aligned} u &= \alpha_1 + \alpha_2 x + \alpha_3 y \\ v &= \alpha_4 + \alpha_5 x + \alpha_6 y \end{aligned} \tag{4.3}$$

The six constants α can be evaluated easily by solving the two sets of three simultaneous equations which will arise if the nodal coordinates are inserted and the displacements equated to the appropriate nodal displacements. Writing, for example,

$$\begin{aligned} u_i &= \alpha_1 + \alpha_2 x_i + \alpha_3 y_i \\ u_j &= \alpha_1 + \alpha_2 x_j + \alpha_3 y_j \\ u_m &= \alpha_1 + \alpha_2 x_m + \alpha_3 y_m \end{aligned} \tag{4.4}$$

we can easily solve for α_1, α_2, and α_3 in terms of the nodal displacements u_i, u_j, u_m and obtain finally

$$u = \frac{1}{2\Delta}[(a_i + b_i x + c_i y)u_i + (a_j + b_j x + c_j y)u_j + (a_m + b_m x + c_m y)u_m] \tag{4.5a}$$

in which

$$\begin{aligned} a_i &= x_j y_m - x_m y_j \\ b_i &= y_j - y_m \\ c_i &= x_m - x_j \end{aligned} \tag{4.5b}$$

with the other coefficients obtained by a cycle permutation of subscripts in the order, i, j, m, and where†

$$2\Delta = \det \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix} = 2 \cdot (\text{area of triangle } ijm) \tag{4.5c}$$

As the equations for the vertical displacement v are similar we also have

$$v = \frac{1}{2\Delta}[(a_i + b_i x + c_i y)v_i + (a_j + b_j x + c_j y)v_j + (a_m + b_m x + c_m y)v_m] \tag{4.6}$$

Though not strictly necessary at this stage we can represent the above relations, Eqs (4.5a) and (4.6), in the standard form of Eq. (2.1):

$$\mathbf{u} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \mathbf{N}\mathbf{a}^e = [\mathbf{I}N_i, \mathbf{I}N_j, \mathbf{I}N_m]\mathbf{a}^e \tag{4.7}$$

with $\mathbf{I}$ a two by two identity matrix, and

$$N_i = \frac{a_i + b_i x + c_i y}{2\Delta}, \qquad \text{etc.} \tag{4.8}$$

The chosen displacement function automatically guarantees continuity of displacement with adjacent elements because the displacements vary linearly along any side of the triangle and, with identical displacement imposed at the nodes, the same displacement will clearly exist all along an interface.

4.2.2 Strain (total)

The total strain at any point within the element can be defined by its three components which contribute to internal work. Thus

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x}, & 0 \\ 0, & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y}, & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} = \mathbf{S}\mathbf{u} \tag{4.9}$$

† *Note*: If coordinates are taken from the centroid of the element then

$$x_i + x_j + x_m = y_i + y_j + y_m = 0 \quad \text{and} \quad a_i = 2\Delta/3 = a_j = a_m$$

See also Appendix D for a summary of integrals for a triangle.

Substituting Eq. (4.7) we have

$$\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{a}^e = [\mathbf{B}_i, \mathbf{B}_j, \mathbf{B}_m] \begin{Bmatrix} \mathbf{a}_i \\ \mathbf{a}_j \\ \mathbf{a}_m \end{Bmatrix} \tag{4.10a}$$

with a typical matrix $\mathbf{B}_i$ given by

$$\mathbf{B}_i = \mathbf{S}N_i = \begin{bmatrix} \dfrac{\partial N_i}{\partial x}, & 0 \\ 0, & \dfrac{\partial N_i}{\partial y} \\ \dfrac{\partial N_i}{\partial y}, & \dfrac{\partial N_i}{\partial x} \end{bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} b_i, & 0 \\ 0, & c_i \\ c_i, & b_i \end{bmatrix} \tag{4.10b}$$

This defines matrix $\mathbf{B}$ of Eq. (2.4) explicitly.

It will be noted that in this case the $\mathbf{B}$ matrix is independent of the position within the element, and hence the strains are constant throughout it. Obviously, the criterion of constant strain mentioned in Chapter 2 is satisfied by the shape functions.

4.2.3 Elasticity matrix

The matrix $\mathbf{D}$ of Eq. (2.5)

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \mathbf{D} \left(\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \boldsymbol{\varepsilon}_0 \right) \tag{4.11}$$

can be explicitly stated for any material (excluding here $\boldsymbol{\sigma}_0$ which is simply additive). To consider the special cases in two dimensions it is convenient to start from the form

$$\boldsymbol{\varepsilon} = \mathbf{D}^{-1}\boldsymbol{\sigma} + \boldsymbol{\varepsilon}_0$$

and impose the conditions of plane stress or plane strain.

Plane stress – isotropic material

For plane stress in an isotropic material we have by definition,

$$\begin{aligned} \varepsilon_x &= \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \varepsilon_{x0} \\ \varepsilon_y &= -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} + \varepsilon_{y0} \\ \gamma_{xy} &= \frac{2(1+\nu)\tau_{xy}}{E} + \gamma_{xy0} \end{aligned} \tag{4.12}$$

Solving the above for the stresses, we obtain the matrix $\mathbf{D}$ as

$$\mathbf{D} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \tag{4.13}$$

and the initial strains as

$$\boldsymbol{\varepsilon}_0 = \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{x0} \\ \gamma_{xy0} \end{Bmatrix} \tag{4.14}$$

in which E is the elastic modulus and ν is Poisson's ratio.

Plane strain – isotropic material

In this case a normal stress σ_z exists in addition to the other three stress components. Thus we now have

$$\begin{aligned} \varepsilon_x &= \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} + \varepsilon_{x0} \\ \varepsilon_y &= -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E} + \varepsilon_{y0} \\ \gamma_{xy} &= \frac{2(1+\nu)\tau_{xy}}{E} + \gamma_{xy0} \end{aligned} \tag{4.15}$$

and in addition

$$\varepsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E} + \varepsilon_{z0} = 0$$

which yields

$$\sigma_z = \nu(\sigma_x + \sigma_y) - E\varepsilon_{z0}$$

On eliminating σ_z and solving for the three remaining stresses we obtain the matrix $\mathbf{D}$ as

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix} \tag{4.16}$$

and the initial strains

$$\boldsymbol{\varepsilon}_0 = \begin{Bmatrix} \varepsilon_{x0} + \nu\varepsilon_{z0} \\ \varepsilon_{y0} + \nu\varepsilon_{z0} \\ \gamma_{xy0} \end{Bmatrix} \tag{4.17}$$

Anisotropic materials

For a completely anisotropic material, 21 independent elastic constants are necessary to define completely the three-dimensional stress–strain relationship.[4,5]

If two-dimensional analysis is to be applicable a symmetry of properties must exist, implying at most six independent constants in the $\mathbf{D}$ matrix. Thus, it is always possible to write

$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ & d_{22} & d_{23} \\ \text{sym.} & & d_{33} \end{bmatrix} \tag{4.18}$$

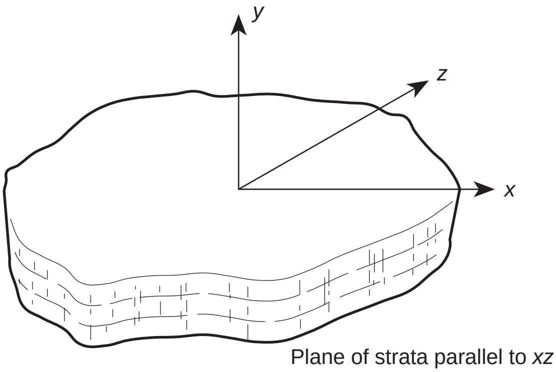


Fig. 4.2 A stratified (transversely isotropic) material.

to describe the most general two-dimensional behaviour. (The necessary symmetry of the **D** matrix follows from the general equivalence of the Maxwell–Betti reciprocal theorem and is a consequence of invariant energy irrespective of the path taken to reach a given strain state.)

A case of particular interest in practice is that of a 'stratified' or transversely isotropic material in which a rotational symmetry of properties exists within the plane of the strata. Such a material possesses only five independent elastic constants.

The general stress–strain relations give in this case, following the notation of Lekhnitskii[4] and taking now the y-axis as perpendicular to the strata (neglecting initial strain) (Fig. 4.2),

$$\begin{aligned}
\varepsilon_x &= \frac{\sigma_x}{E_1} - \frac{\nu_2 \sigma_y}{E_2} - \frac{\nu_1 \sigma_z}{E_1} \\
\varepsilon_y &= -\frac{\nu_2 \sigma_x}{E_2} + \frac{\sigma_y}{E_2} - \frac{\nu_2 \sigma_z}{E_2} \\
\varepsilon_z &= -\frac{\nu_1 \sigma_x}{E_1} - \frac{\nu_2 \sigma_y}{E_2} + \frac{\sigma_z}{E_1} \\
\gamma_{xz} &= \frac{2(1+\nu_1)}{E_1} \tau_{xz} \\
\gamma_{xy} &= \frac{1}{G_2} \tau_{xy} \\
\gamma_{yz} &= \frac{1}{G_2} \tau_{yz}
\end{aligned} \tag{4.19}$$

in which the constants E_1, ν_1 (G_1 is dependent) are associated with the behaviour in the plane of the strata and E_2, G_2, ν_2 with a direction normal to the plane.

The **D** matrix in two dimensions now becomes, taking $E_1/E_2 = n$ and $G_2/E_2 = m$,

$$\mathbf{D} = \frac{E_2}{1 - n\nu_2^2} \begin{bmatrix} n & n\nu_2 & 0 \\ n\nu_2 & 1 & 0 \\ 0 & 0 & m(1 - n\nu_2^2) \end{bmatrix} \tag{4.20}$$

for plane stress or

$$\mathbf{D} = \frac{E_2}{(1+\nu_1)(1-\nu_1-2n\nu_2^2)} \times \begin{bmatrix} n(1-n\nu_2^2) & n\nu_2(1+\nu_1) & 0 \\ n\nu_2(1+\nu_1) & (1-\nu_1^2) & 0 \\ 0 & 0 & m(1+\nu_1)(1-\nu_1-2n\nu_2^2) \end{bmatrix} \tag{4.21}$$

for plane strain.

When, as in Fig. 4.3, the direction of the strata is inclined to the x-axis then to obtain the $\mathbf{D}$ matrices in universal coordinates a transformation is necessary. Taking $\mathbf{D}'$ as relating the stresses and strains in the inclined coordinate system (x', y') it is easy to show that

$$\mathbf{D} = \mathbf{T}\mathbf{D}'\mathbf{T}^{\mathrm{T}} \tag{4.22}$$

where

$$\mathbf{T} = \begin{bmatrix} \cos^2\beta & \sin^2\beta & -2\sin\beta\cos\beta \\ \sin^2\beta & \cos^2\beta & 2\sin\beta\cos\beta \\ \sin\beta\cos\beta & -\sin\beta\cos\beta & \cos^2\beta-\sin^2\beta \end{bmatrix} \tag{4.23}$$

with β as defined in Fig. 4.3.

If the stress systems $\boldsymbol{\sigma}'$ and $\boldsymbol{\sigma}$ correspond to $\boldsymbol{\varepsilon}'$ and $\boldsymbol{\varepsilon}$ respectively then by equality of work

$$\boldsymbol{\sigma}'^{\mathrm{T}}\boldsymbol{\varepsilon}' = \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\varepsilon}$$

or

$$\boldsymbol{\varepsilon}'^{\mathrm{T}}\mathbf{D}'\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\varepsilon}$$

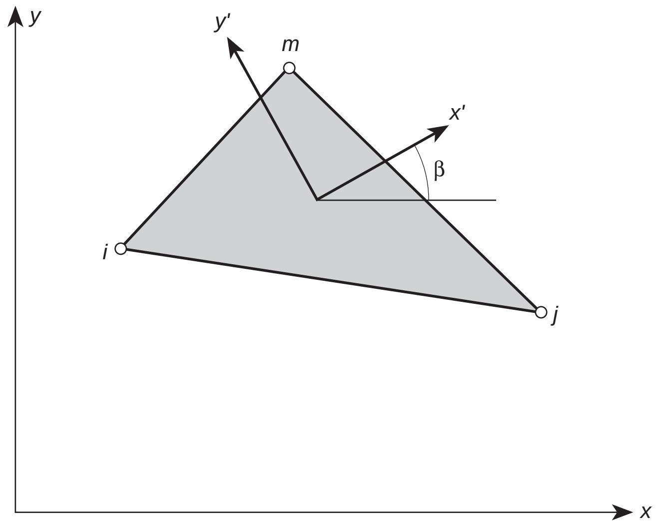


Fig. 4.3 An element of a stratified (transversely isotropic) material.

from which Eq. (4.22) follows on noting (see also Chapter 1)

$$\boldsymbol{\varepsilon}' = \mathbf{T}^{\mathrm{T}}\boldsymbol{\varepsilon} \tag{4.24}$$

4.2.4 Initial strain (thermal strain)

'Initial' strains, i.e., strains which are independent of stress, may be due to many causes. Shrinkage, crystal growth, or, most frequently, temperature change will, in general, result in an initial strain vector:

$$\boldsymbol{\varepsilon}_0 = [\varepsilon_{x0} \quad \varepsilon_{y0} \quad \varepsilon_{z0} \quad \gamma_{xy0} \quad \gamma_{yz0} \quad \gamma_{zx0}]^{\mathrm{T}} \tag{4.25}$$

Although this initial strain may, in general, depend on the position within the element, it will here be defined by average, constant values to be consistent with the constant strain conditions imposed by the prescribed displacement function.

For an isotropic material in an element subject to a temperature rise θ^e with a coefficient of thermal expansion α we will have

$$\boldsymbol{\varepsilon}_0 = \alpha\theta^e[1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0]^{\mathrm{T}} \tag{4.26}$$

as no shear strains are caused by a thermal dilatation. Thus, for plane stress, Eq. (4.14) yields the initial strains given by

$$\boldsymbol{\varepsilon}_0 = \alpha\theta^e \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} = \alpha\theta^e\mathbf{m} \tag{4.27}$$

In *plane strain* the σ_z stress perpendicular to the xy plane will develop due to the thermal expansion as shown above. Using Eq. (4.17) the initial thermal strains for this case are given by

$$\boldsymbol{\varepsilon}_0 = (1+\nu)\alpha\theta^e\mathbf{m} \tag{4.28}$$

Anisotropic materials present special problems, since the coefficients of thermal expansion may vary with direction. In the general case the thermal strains are given by

$$\boldsymbol{\varepsilon}_0 = \boldsymbol{\alpha}\theta^e \tag{4.29}$$

where $\boldsymbol{\alpha}$ has properties similar to strain. Accordingly, it is always possible to find orthogonal directions for which $\boldsymbol{\alpha}$ is diagonal. If we let x' and y' denote the principal thermal directions of the material, the initial strain due to thermal expansion for a plane stress state becomes (assuming z' is a principal direction)

$$\boldsymbol{\varepsilon}' = \theta^e \begin{Bmatrix} \varepsilon_{x'0} \\ \varepsilon_{y'0} \\ \gamma_{x'y'0} \end{Bmatrix} = \theta^e \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{Bmatrix} \tag{4.30}$$

where α_1 and α_2 are the expansion coefficients referred to the x' and y' axes, respectively.

To obtain strain components in the x, y system it is necessary to use the strain transformation

$$\boldsymbol{\varepsilon}_0' = \mathbf{T}^{\mathrm{T}} \boldsymbol{\varepsilon}_0 \tag{4.31}$$

where $\mathbf{T}$ is again given by Eq. (4.23). Thus, $\boldsymbol{\varepsilon}_0$ can be simply evaluated. It will be noted that the shear component of strain is no longer equal to zero in the x, y coordinates.

4.2.5 The stiffness matrix

The stiffness matrix of the element ijm is defined from the general relationship (2.13) with the coefficients

$$\mathbf{K}_{ij}^e = \int \mathbf{B}_i^{\mathrm{T}} \mathbf{D} \mathbf{B}_j t \, \mathrm{d}x \, \mathrm{d}y \tag{4.32}$$

where t is the thickness of the element and the integration is taken over the area of the triangle. If the thickness of the element is assumed to be constant, an assumption convergent to the truth as the size of elements decreases, then, as neither of the matrices contains x or y we have simply

$$\mathbf{K}_{ij}^e = \mathbf{B}_i^{\mathrm{T}} \mathbf{D} \mathbf{B}_j t \Delta \tag{4.33}$$

where Δ is the area of the triangle [already defined by Eq. (4.5)]. This form is now sufficiently explicit for computation with the actual matrix operations being left to the computer.

4.2.6 Nodal forces due to initial strain

These are given directly by the expression Eq. (2.13b) which, on performing the integration, becomes

$$(\mathbf{f}_i)_{\varepsilon_0}^e = -\mathbf{B}_i^{\mathrm{T}} \mathbf{D} \boldsymbol{\varepsilon}_0 t \Delta, \quad \text{etc.} \tag{4.34}$$

These 'initial strain' forces contribute to the nodes of an element in an unequal manner and require precise evaluation. Similar expressions are derived for initial stress forces.

4.2.7 Distributed body forces

In the general case of plane stress or strain each element of unit area in the xy plane is subject to forces

$$\mathbf{b} = \begin{Bmatrix} b_x \\ b_y \end{Bmatrix}$$

in the direction of the appropriate axes.

Again, by Eq. (2.13b), the contribution of such forces to those at each node is given by

$$\mathbf{f}_i^e = -\int N_i \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} \mathrm{d}x\,\mathrm{d}y$$

or by Eq. (4.7),

$$\mathbf{f}_i^e = -\begin{Bmatrix} b_x \\ b_y \end{Bmatrix} \int N_i \,\mathrm{d}x\,\mathrm{d}y, \quad \text{etc.} \tag{4.35}$$

if the body forces b_x and b_y are constant. As N_i is not constant the integration has to be carried out explicitly. Some general integration formulae for a triangle are given in Appendix D.

In this special case the calculation will be simplified if the origin of coordinates is taken at the centroid of the element. Now

$$\int x \,\mathrm{d}x\,\mathrm{d}y = \int y \,\mathrm{d}x\,\mathrm{d}y = 0$$

and on using Eq. (4.8)

$$\mathbf{f}_i^e = -\begin{Bmatrix} b_x \\ b_y \end{Bmatrix} \int \frac{a_i \,\mathrm{d}x\,\mathrm{d}y}{2\Delta} = -\begin{Bmatrix} b_x \\ b_y \end{Bmatrix} \frac{a_i}{2} = -\begin{Bmatrix} b_x \\ b_y \end{Bmatrix} \frac{\Delta}{3} \tag{4.36}$$

by relations noted on page 89.

Explicitly, for the whole element

$$\mathbf{f}^e = \begin{Bmatrix} \mathbf{f}_i^e \\ \mathbf{f}_j^e \\ \mathbf{f}_m^e \end{Bmatrix} = -\begin{Bmatrix} b_x \\ b_y \\ b_x \\ b_y \\ b_x \\ b_y \end{Bmatrix} \frac{\Delta}{3} \tag{4.37}$$

which means simply that the total forces acting in the x and y directions due to the body forces are distributed to the nodes in three equal parts. This fact corresponds with physical intuition, and was often assumed implicitly.

4.2.8 Body force potential

In many cases the body forces are defined in terms of a body force potential ϕ as

$$b_x = -\frac{\partial \phi}{\partial x} \qquad b_y = -\frac{\partial \phi}{\partial y} \tag{4.38}$$

and this potential, rather than the values of b_x and b_y, is known throughout the region and is specified at nodal points. If $\boldsymbol{\phi}^e$ lists the three values of the potential associated with the nodes of the element, i.e.,

$$\boldsymbol{\phi}^e = \begin{Bmatrix} \phi_i \\ \phi_j \\ \phi_m \end{Bmatrix} \tag{4.39}$$

and has to correspond with constant values of b_x and b_y, ϕ must vary linearly within the element. The ‘shape function’ of its variation will obviously be given by a procedure identical to that used in deriving Eqs (4.4)–(4.6), and yields

$$\phi = [N_i, N_j, N_m]\boldsymbol{\phi}^e \tag{4.40}$$

Thus,

$$b_x = -\frac{\partial \phi}{\partial x} = -[b_i, b_j, b_m]\frac{\boldsymbol{\phi}^e}{2\Delta}$$

and

$$b_y = -\frac{\partial \phi}{\partial y} = -[c_i, c_j, c_m]\frac{\boldsymbol{\phi}^e}{2\Delta} \tag{4.41}$$

The vector of nodal forces due to the body force potential will now replace Eq. (4.37) by

$$\mathbf{f}^e = \frac{1}{6}\begin{bmatrix} b_i, & b_j, & b_m \\ c_i, & c_j, & c_m \\ b_i, & b_j, & b_m \\ c_i, & c_j, & c_m \\ b_i, & b_j, & b_m \\ c_i, & c_j, & c_m \end{bmatrix}\boldsymbol{\phi}^e \tag{4.42}$$

4.2.9 Evaluation of stresses

The derived formulae enable the full stiffness matrix of the structure to be assembled, and a solution for displacements to be obtained.

The stress matrix given in general terms in Eq. (2.16) is obtained by the appropriate substitutions for each element.

The stresses are, by the basic assumption, constant within the element. It is usual to assign these to the centroid of the element, and in most of the examples in this chapter this procedure is followed. An alternative consists of obtaining stress values at the nodes by averaging the values in the adjacent elements. Some ‘weighting’ procedures have been used in this context on an empirical basis but their advantage appears small.

It is also usual to calculate the principal stresses and their directions for every element. In Chapter 14 we shall return to the problem of stress recovery and show that better procedures of stress recovery exist.[6,7]

4.3 Examples – an assessment of performance

There is no doubt that the solution to plane elasticity problems as formulated in Sec. 4.2 is, in the limit of subdivision, an exact solution. Indeed at any stage of a finite subdivision it is an approximate solution as is, say, a Fourier series solution with a limited number of terms.

As explained in Chapter 2, the total strain energy obtained during any stage of approximation will be below the true strain energy of the exact solution. In practice it will mean that the displacements, and hence also the stresses, will be underestimated by the approximation in its *general picture*. However, it must be emphasized that this is not necessarily true at every point of the continuum individually; hence the value of such a bound in practice is not great.

What is important for the engineer to know is the order of accuracy achievable in typical problems with a certain fineness of element subdivision. In any particular case the error can be assessed by comparison with known, exact, solutions or by a study of the convergence, using two or more stages of subdivision.

With the development of experience the engineer can assess *a priori* the order of approximation that will be involved in a specific problem tackled with a given element subdivision. Some of this experience will perhaps be conveyed by the examples considered in this book.

In the first place attention will be focused on some simple problems for which exact solutions are available.

4.3.1 Uniform stress field

If the exact solution is in fact that of a uniform stress field then, whatever the element subdivision, the finite element solution will coincide exactly with the exact one. This is an obvious corollary of the formulation; nevertheless it is useful as a first check of written computer programs.

4.3.2 Linearly varying stress field

Here, obviously, the basic assumption of constant stress within each element means that the solution will be approximate only. In Fig. 4.4 a simple example of a beam subject to constant bending moment is shown with a fairly coarse subdivision. It is readily seen that the axial (σ_y) stress given by the element 'straddles' the exact values and, in fact, if the constant stress values are associated with centroids of the elements and plotted, the best 'fit' line represents the exact stresses.

The horizontal and shear stress components differ again from the exact values (which are simply zero). Again, however, it will be noted that they oscillate by equal, small amounts around the exact values.

At internal nodes, if the average of the stresses of surrounding elements is taken it will be found that the exact stresses are very closely represented. The average at external faces is not, however, so good. The overall improvement in representing the stresses by nodal averages, as shown in Fig. 4.4, is often used in practice for contour plots. However, we shall show in Chapter 14 a method of recovery which gives much improved values at both interior and boundary points.

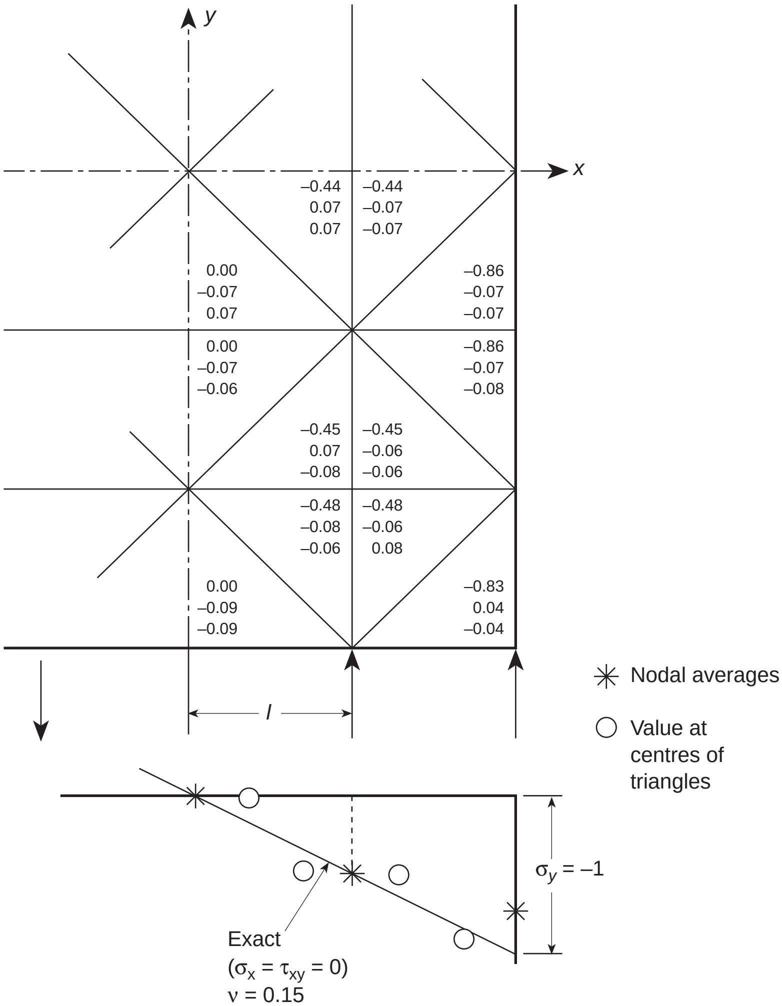


Fig. 4.4 Pure bending of a beam solved by a coarse subdivision into elements of triangular shape. (Values of σ_y, σ_x, and τ_{xy} listed in that order.)

4.3.3 Stress concentration

A more realistic test problem is shown in Figs 4.5 and 4.6. Here the flow of stress around a circular hole in an isotropic and in an anisotropic stratified material is considered when the stress conditions are uniform.[8] A graded division into elements is used to allow a more detailed study in the region where high stress gradients are expected. The accuracy achievable can be assessed from Fig. 4.6 where some of the results are compared against exact solutions.[3,9]

In later chapters we shall see that even more accurate answers can be obtained with the use of more elaborate elements; however, the principles of the analysis remain identical.

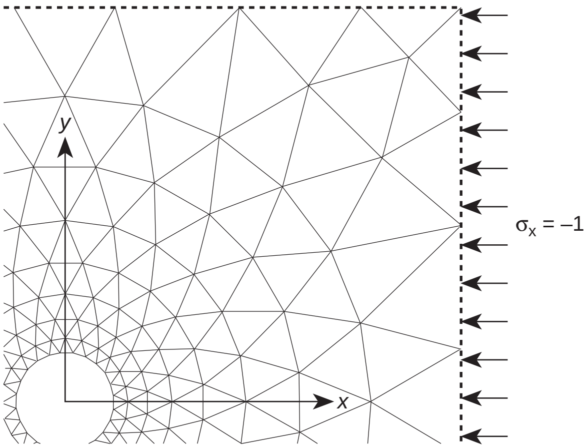


Fig. 4.5 A circular hole in a uniform stress field: (a) isotropic material; (b) stratified (orthotropic) material; $E_x = E_1 = 1$, $E_y = E_2 = 3$, $\nu_1 = 0.1$, $\nu_2 = 0$, $G_{xy} = 0.42$.

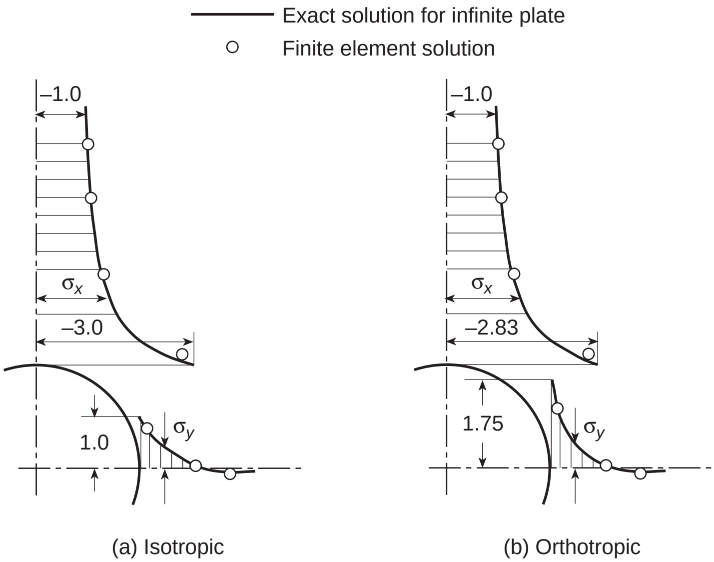


Fig. 4.6 Comparison of theoretical and finite element results for cases (a) and (b) of Fig. 4.5.

4.4 Some practical applications

Obviously, the practical applications of the method are limitless, and the finite element method has superseded experimental technique for plane problems because of its high accuracy, low cost, and versatility. The ease of treatment of material anisotropy, thermal stresses, or body force problems add to its advantages.

A few examples of actual early applications of the finite element method to complex problems of engineering practice will now be given.

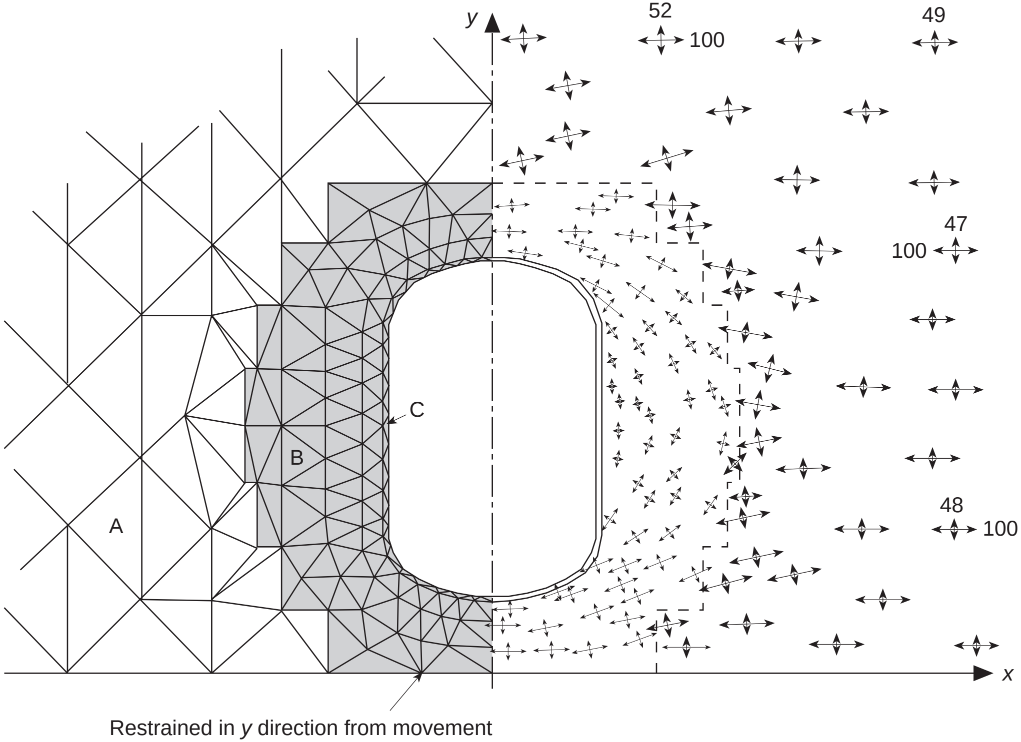


Fig. 4.7 A reinforced opening in a plate. Uniform stress field at a distance from opening $\sigma_x = 100$, $\sigma_y = 50$. Thickness of plate regions A, B, and C is in the ratio of 1 : 3 : 23.

4.4.1 Stress flow around a reinforced opening (Fig. 4.7)

In steel pressure vessels or aircraft structures, openings have to be introduced in the stressed skin. The penetrating duct itself provides some reinforcement round the edge and, in addition, the skin itself is increased in thickness to reduce the stresses due to concentration effects.

Analysis of such problems treated as cases of plane stress present no difficulties. The elements are chosen so as to follow the thickness variation, and appropriate values of this are assigned.

The narrow band of thick material near the edge can be represented either by special bar-type elements, or by very thin triangular elements of the usual type, to which appropriate thickness is assigned. The latter procedure was used in the problem shown in Fig. 4.7 which gives some of the resulting stresses near the opening itself. The fairly large extent of the region introduced in the analysis and the grading of the mesh should be noted.

4.4.2 An anisotropic valley subject to techtonic stress[8] (Fig. 4.8)

A symmetrical valley subject to a uniform horizontal stress is considered. The material is stratified, and hence is 'transversely isotropic', and the direction of strata varies from point to point.

The stress plot shows the tensile region that develops. This phenomenon is of considerable interest to geologists and engineers concerned with rock mechanics. (See reference 10 for additional applications on this topic.)

4.4.3 A dam subject to external and internal water pressures[11,12]

A buttress dam on a somewhat complex rock foundation is shown in Fig. 4.9 and analysed. This dam (completed in 1964) is of particular interest as it is the first to which the finite element method was applied during the design stage. The heterogeneous foundation region is subject to plane strain conditions while the dam itself is considered in a state of plane stress of variable thickness.

With external and gravity loading no special problems of analysis arise.

When pore pressures are considered, the situation, however, requires perhaps some explanation.

It is well known that in a porous material the water pressure is transmitted to the structure as a *body force* of magnitude

$$b_x = -\frac{\partial p}{\partial x} \qquad b_y = -\frac{\partial p}{\partial y} \tag{4.43}$$

and that now the external pressure need not be considered.

The pore pressure p is, in fact, now a body force potential, as defined in Eq. (4.38). Figure 4.9 shows the element subdivision of the region and the outline of the dam. Figure 4.10(a) and (b) shows the stresses resulting from gravity (applied to the dam only) and due to water pressure assumed to be acting as an external load or, alternatively,

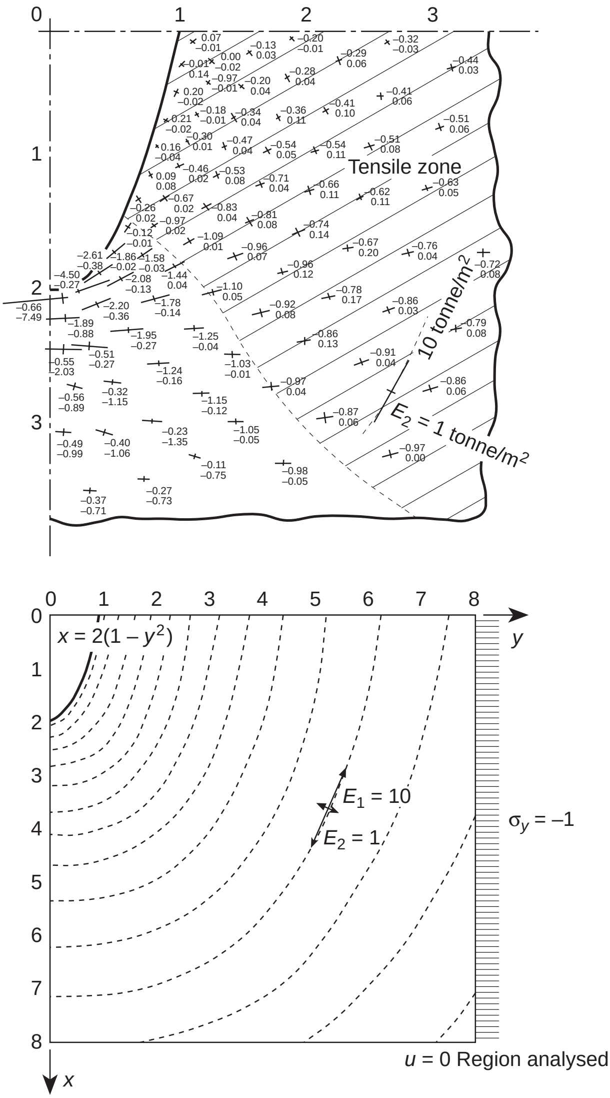


Fig. 4.8 A valley with curved strata subject to a horizontal tectonic stress (plane strain 170 nodes, 298 elements).

as an internal pore pressure. Both solutions indicate large tensile regions, but the increase of stresses due to the second assumption is important.

The stresses calculated here are the so-called ‘effective’ stresses. These represent the forces transmitted between the solid particles and are defined in terms of the *total* stresses $\boldsymbol{\sigma}$ and the pore pressures p by

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} + \mathbf{m}p \qquad \mathbf{m}^{\mathrm{T}} = [1, 1, 0] \tag{4.44}$$

i.e., simply by removing the hydrostatic pressure component from the *total* stress.[10,13]

The effective stress is of particular importance in the mechanics of porous media such as those that occur in the study of soils, rocks, or concrete. The basic assumption

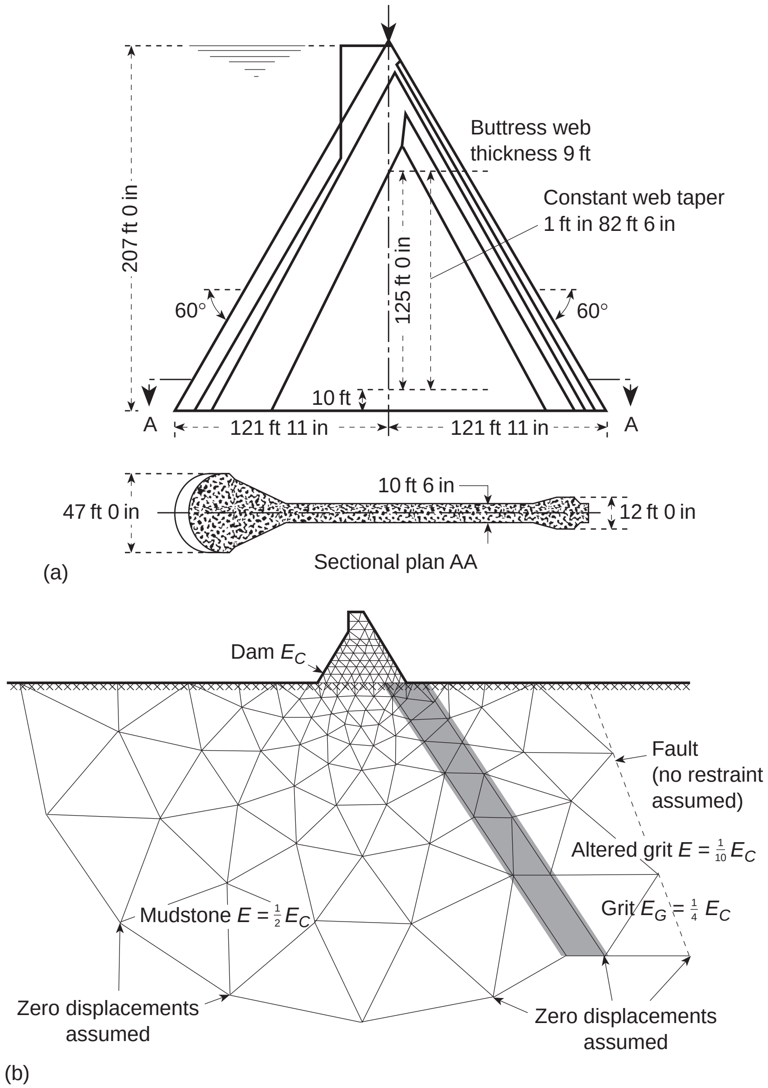
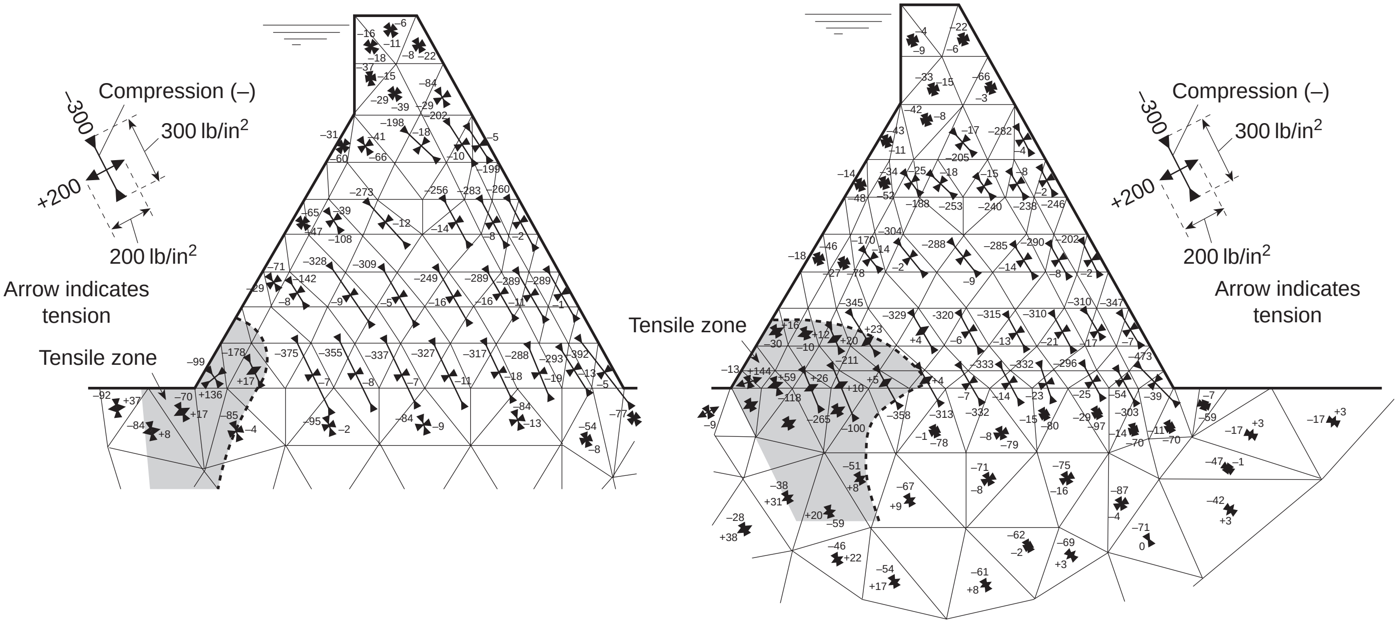


Fig. 4.9 Stress analysis of a buttress dam. A plane stress condition is assumed in the dam and plane strain in the foundation. (a) The buttress section analysed. (b) Extent of foundation considered and division into finite elements.

in deriving the body forces of Eq. (4.43) is that only the effective stress is of any importance in deforming the solid phase. This leads immediately to another possibility of formulation.[14] If we examine the equilibrium conditions of Eq. (2.10) we note that this is written in terms of total stresses. Writing the constitutive relation, Eq. (2.5), in terms of effective stresses, i.e.,

$$\boldsymbol{\sigma}' = \mathbf{D}'(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0) + \boldsymbol{\sigma}'_0 \tag{4.45}$$

and substituting into the equilibrium equation (2.10) we find that Eq. (2.12) is again obtained, with the stiffness matrix using the matrix $\mathbf{D}'$ and the force terms of



Below the foundation initial rock stresses should be superimposed

(a) (b)

Fig. 4.10 Stress analysis of the buttress dam of Fig. 4.9. Principal stresses for gravity loads are combined with water pressures, which are assumed to act (a) as external loads, (b) as body forces due to pore pressure.

Eq. (2.13b) being augmented by an additional force

$$-\int_{V^e} \mathbf{B}^{\mathrm{T}} \mathbf{m} p \, \mathrm{d(vol)} \tag{4.46}$$

or, if p is interpolated by shape functions N'_i, the force becomes

$$-\int_{V^e} \mathbf{B}^{\mathrm{T}} \mathbf{m} \mathbf{N}' \, \mathrm{d(vol)} \bar{\mathbf{p}}^e \tag{4.47}$$

This alternative form of introducing pore pressure effects allows a discontinuous interpolation of p to be used [as in Eq. (4.46) no derivatives occur] and this is now frequently used in practice.

4.4.4 Cracking

The tensile stresses in the previous example will doubtless cause the rock to crack. If a stable situation can develop when such a crack spreads then the dam can be considered safe.

Cracks can be introduced very simply into the analysis by assigning zero elasticity values to chosen elements. An analysis with a wide cracked wedge is shown in Fig. 4.11, where it can be seen that with the extent of the crack assumed no tension within the dam body develops.

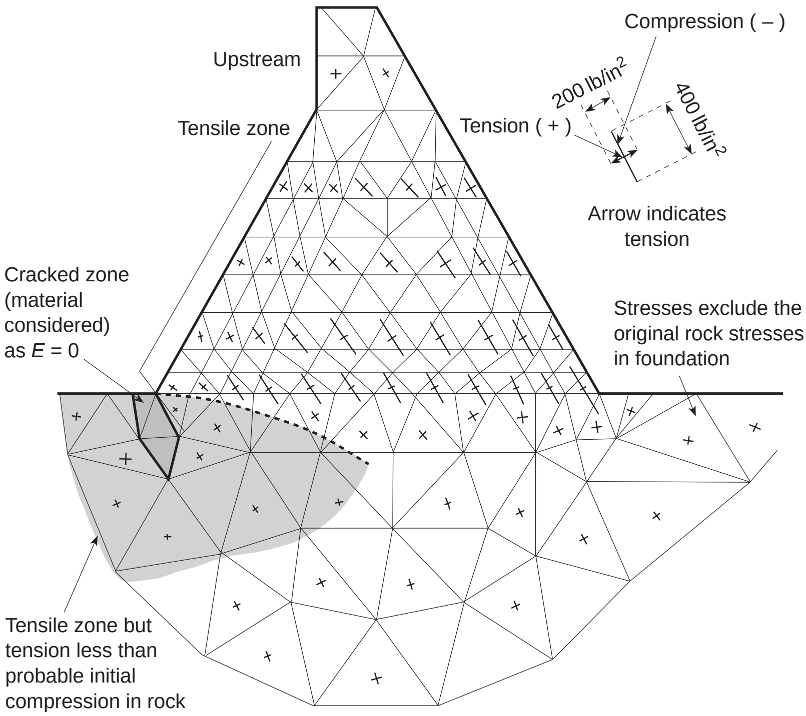


Fig. 4.11 Stresses in a buttress dam. The introduction of a 'crack' modifies the stress distribution [same loading as Fig. 4.10(b)].

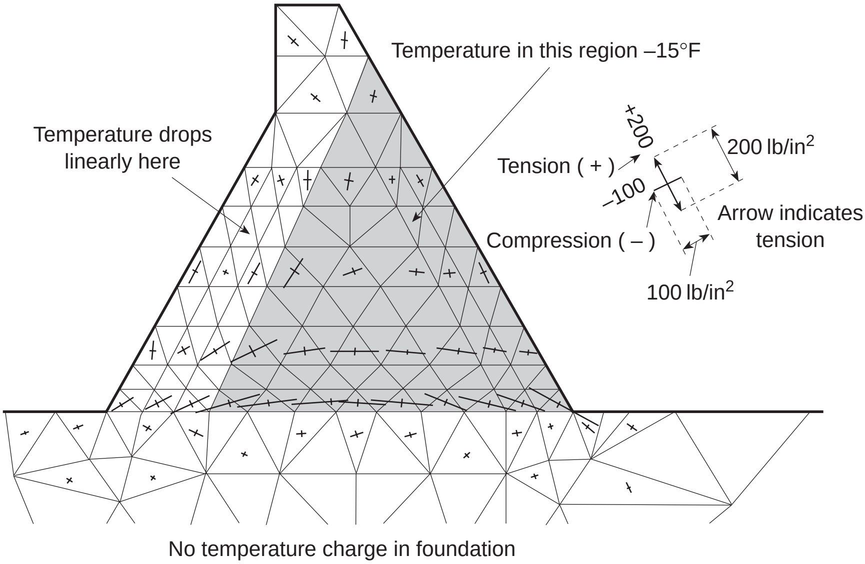


Fig. 4.12 Stress analysis of a buttress dam. Thermal stresses due to cooling of the shaded area by 15 °F ($E = 3 \times 10^6$ lb/in², $\alpha = 6 \times 10^{-6}$/°F).

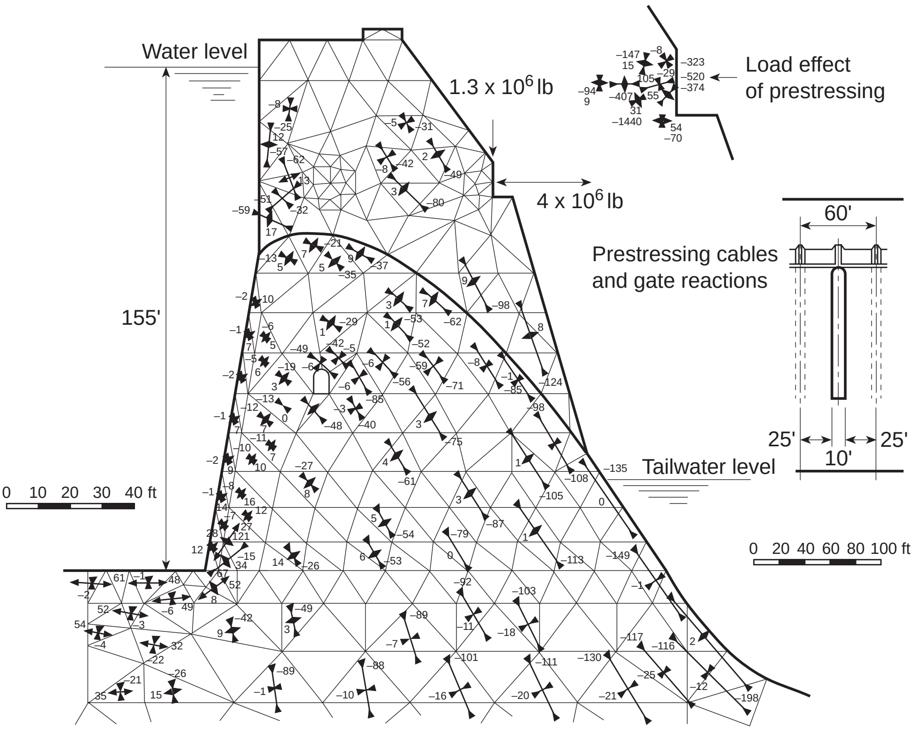


Fig. 4.13 A large barrage with piers and prestressing cables.

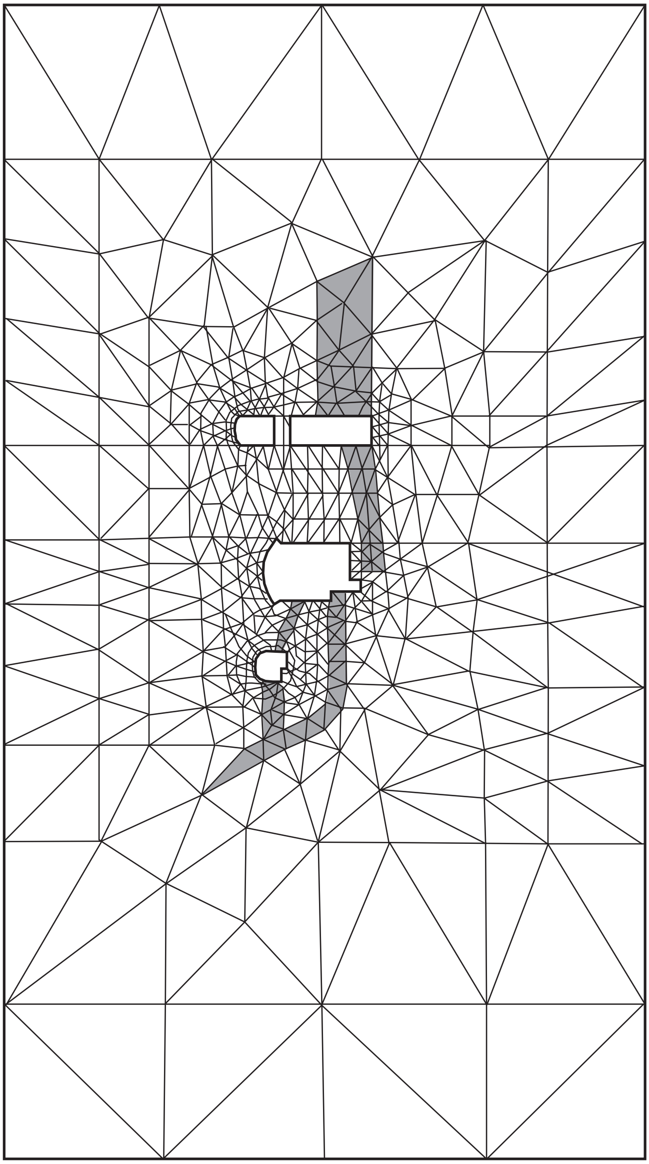


Fig. 4.14 An underground power station. Mesh used in analysis.

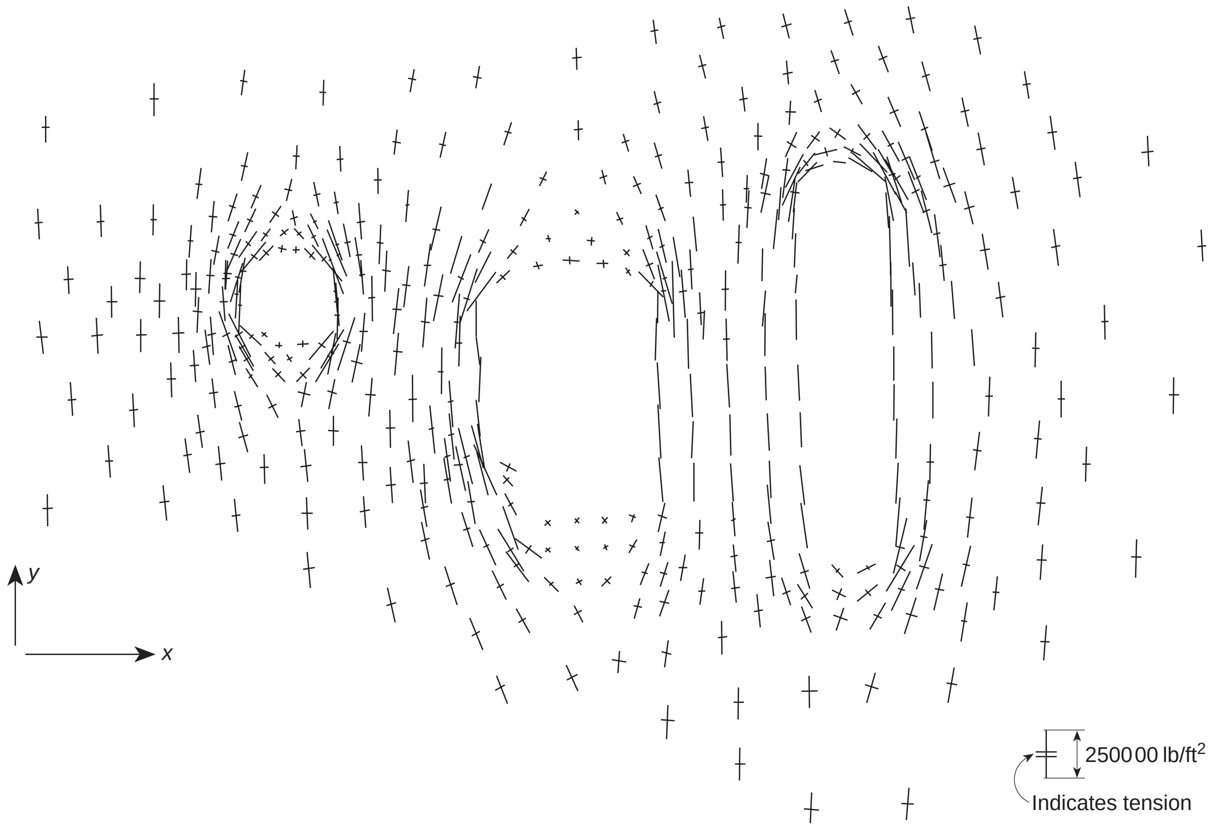


Fig. 4.15 An underground power station. Plot of principal stresses.

A more elaborate procedure for allowing crack propagation and resulting stress redistribution can be developed (see Volume 2).

4.4.5 Thermal stresses

As an example of thermal stress computation the same dam is shown under simple temperature distribution assumptions. Results of this analysis are given in Fig. 4.12.

4.4.6 Gravity dams

A buttress dam is a natural example for the application of finite element methods. Other types, such as gravity dams with or without piers and so on, can also be simply treated. Figure 4.13 shows an analysis of a large dam with piers and crest gates.

In this case the approximation of assuming a two-dimensional treatment in the vicinity of the abrupt change of section, i.e., where the piers join the main body of the dam, is clearly involved, but this leads to localized errors only.

It is important to note here how, in a single solution, the grading of element size is used to study concentration of stress at the cable anchorages, the general stress flow in the dam, and the foundation behaviour. The linear ratio of size of largest to smallest elements is of the order of 30 to 1 (the largest elements occurring in the foundation are not shown in the figure).

4.4.7 Underground power station

This last example, illustrated in Figs 4.14 and 4.15, shows an interesting application. Here principal stresses are plotted automatically. In this analysis many different components of $\boldsymbol{\sigma}_0$, the initial stress, were used due to uncertainty of knowledge about geological conditions. The rapid solution and plot of many results enabled the limits within which stresses vary to be found and an engineering decision arrived at. In this example, the exterior boundaries were taken far enough and 'fixed' ($u = v = 0$). However, a better treatment could be made using infinite elements as described in Sec. 9.13.

4.5 Special treatment of plane strain with an incompressible material

It will have been noted that the relationship (4.16) defining the elasticity matrix $\mathbf{D}$ for an isotropic material breaks down when Poisson's ratio reaches a value of 0.5 as the factor in parentheses becomes infinite. A simple way of side-stepping this difficulty is to use values of Poisson's ratio approximating to 0.5 but not equal to it. Experience shows, however, that if this is done the solution deteriorates unless special formulations such as those discussed in Chapter 12 are used.

4.6 Concluding remark

In subsequent chapters, we shall introduce elements which give much greater accuracy for the same number of degrees of freedom in a particular problem. This has led to the belief that the simple triangle used here is completely superseded. In recent years, however, its very simplicity has led to its revival in practical use in combination with the error estimation and adaptive procedures discussed in Chapters 14 and 15.

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